In É. Baudrier, B. Naegel, A. Krähenbühl, M. Tajine (eds.), Discrete Geometry and Mathematical Morphology (DGMM 2022), Lecture Notes in Computer Science, vol. 13493, pp. 193–205, Springer, Cham, 2022. — The final publication is available at link.springer.com.

Equivariance-Based Analysis of PDE Evolutions Related to Multivariate Medians

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Abstract. For multivariate data there exist several concepts generalising the median, which differ by their equivariance properties w.r.t. transformations of the data space (e.g. Euclidean, affine). In earlier work on the asymptotic analysis of multivariate image filters built upon these concepts, it was observed that several affine equivariant median filters approximate the same system of partial differential equations (PDEs). In this paper we discuss the equivariance properties of multivariate medians and their associated PDEs in more detail. We discuss what equivariance concept is the preferable generalisation of the very strong equivariance of the scalar-valued median (sometimes also denoted as morphological equivariance) w.r.t. arbitrary monotone transformation. Moreover, we derive multivariate PDE evolutions systematically from equivariance properties. It turns out that the approximation of the same PDE system by different affine equivariant medians is no coincidence but a necessary implication of their equivariance properties. As a by-product, a more general class of multivariate PDE evolutions with favourable equivariance properties arises.
Keywords: multivariate images – partial differential equations – multivariate median – affine equivariance – morphological filters

1 Introduction

Curvature motion can be described in numerous ways. First of all, it is a curve evolution that can be used for contours or shapes, and which can be stated e.g. by a partial differential equation (PDE) for parametrised curves, and which is a gradient descent for the curve length functional [1]. In a grey-value image, its simultaneous application to all level lines gives rise to an image evolution which can be described by a PDE acting directly on the intensities [9]. As such, it can be used for structure-preserving image simplification. Moreover, it is closely related to median filtering: As proven in [4], space-continuous median filtering with a disc-shaped structuring element of radius ϱ asymptotically approximates curvature motion up to evolution time $\varrho^2/6$. Remarkably, both curvature motion and the median filter are equivariant under arbitrary monotone intensity rescalings, i.e. their application commutes with such rescalings. As they share this strong property with a larger set of fundamental morphological operations, this property is also often called morphological equivariance (or invariance).

In earlier work, several steps have been undertaken to generalise this framework to multivariate (such as colour) images. Regarding median filtering, this requires a generalisation of the median concept to multivariate data for which different proposals have been made in literature since the beginnings of the 20th century, see e.g. [3, 7, 8, 10, 13] and the overview in [11]. Among the differences between these definitions, equivariance properties with regard to transformations of the data space play an important role as they are decisive for the applicability of the concepts to particular categories of data. Applications for the median filtering of multivariate images can be found e.g. in [6, 12, 19–21].

Asymptotic PDE approximation results for bivariate and multivariate median filtering have been presented in [14, 16, 17], see also extensions to adaptive median filtering with morphological amoebas as structuring elements [15]. A remarkable observation was that affine equivariant multivariate medians, despite not coinciding as such, consistently led to the same PDE evolutions, which suggests that common underlying principles of the PDE evolutions themselves related to equivariance can be worth considering. This is the purpose of the present contribution.

Our contribution. We start by discussing the equivariance properties of multivariate medians and their associated PDEs. Referring to the morphological equivariance of the scalar-valued median and curvature motion PDE, we also address the question what is the best multivariate generalisation of that concept.

We then turn to derive multivariate PDE evolutions in a principled way from equivariance properties modelled after multivariate median concepts. In fact, the asymptotic approximation of the same PDE evolution by an entire class of affine equivariant multivariate median filters turns out to be necessary rather than just coincidential. For Euclidean equivariance, partial results are obtained. Considering slightly relaxed requirements, we find a more general class of multivariate PDE evolutions which deserve further study.

Structure of the paper. The remainder of the paper is organised as follows. In Section 2 we recall multivariate median concepts from literature. We collect known facts about their equivariance properties. At the end of the section, we discuss what is the proper counterpart of morphological equivariance in the case of multivariate data. Section 3 lists existing results on the asymptotic approximation of PDEs by multivariate median filters, emphasising the role of equivariance properties in their derivation. In Section 4 we present the systematic direct derivation of bivariate image filtering PDEs from equivariance properties, culminating in a re-derivation of the PDE system associated with affine equivariant multivariate medians. Section 5 illustrates the theoretical findings by numerical examples of PDE evolutions. A short summary in Section 6 ends the paper.

2 Medians and Equivariance

In the following we shortly recall some definitions of multivariate medians and discuss their equivariance properties. Throughout this section we assume that $\mathcal{X}$ is a finite multiset of values $\boldsymbol{x} \in \mathbb{R}^d$, $d \geq 2$.

2.1 Multivariate Medians

The $\boldsymbol{L^1}$ **median** of $\mathcal{X}$ is defined as the point $\boldsymbol{\mu} \in \mathbb{R}^d$ that minimises the sum of Euclidean distances $|\boldsymbol{\mu} - \boldsymbol{x}|$ to all given points $\boldsymbol{x} \in \mathcal{X}$. Having been introduced in 1909 [13], this is the most widespread concept of multivariate median which has been intensively studied since and has also been used in image processing [6, 12, 19–21]. We remark that in exceptional situations (namely, if all data points are collinear, and $\mathcal{X}$ has even cardinality), the L^1 median is non-unique (set-valued) but do not detail this further as it is generally not relevant for our further investigation. Also the following multivariate medians can be set-valued in certain configurations which we will not detail further.

Oja's simplex median [7] instead defines the median as the $\boldsymbol{\mu} \in \mathbb{R}^d$ that minimises the sum of simplex volumes $|[\boldsymbol{\mu}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_d]|$ for all d-tuples of data points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_d \in \mathcal{X}$. Especially in the bivariate case $d = 2$ this means to minimise a sum of triangle areas. Note that we denote by $[\ldots]$ the oriented simplex volume.

To avoid the high computational expense of the Oja median caused by the combinatorial complexity of its definition, [8] proposed the **transformation-retransformation $\boldsymbol{L^1}$ median (TR-$\boldsymbol{L^1}$ median)**, see also [5]. This median is computed by first applying an affine transform T to $\mathcal{X}$ to normalise the data points such that their covariance matrix becomes the $d \times d$ identity matrix, then applying the L^1 median and then applies the inverse transform T^{-1} to yield the final median $\boldsymbol{\mu} \in \mathbb{R}^d$. If all $\boldsymbol{x} \in \mathcal{X}$ lie in a common affine subspace of $\mathbb{R}^d$, special consideration is needed such as applying the procedure in the subspace only.

The **half-space median** [10] is the point $\boldsymbol{\mu} \in \mathcal{X}$ of maximal half-space depth w.r.t. $\mathcal{X}$. Here, the half-space depth is the minimum over all hyperplanes $H \ni \boldsymbol{\mu}$ of the number of points $\boldsymbol{x} \in \mathcal{X}$ that lie on one side of H. Parametrising hyperplanes with unit normal vectors $\boldsymbol{n} \perp H$ this can be expressed as $\boldsymbol{\mu} = \operatorname*{argmax}_{\boldsymbol{\mu} \in \mathbb{R}^d} \min_{\boldsymbol{n} \in \mathbb{R}^d, |\boldsymbol{n}|=1} \sum_{\boldsymbol{x} \in \boldsymbol{\mathcal{X}}} \operatorname{sgn}\langle \boldsymbol{x} - \boldsymbol{\mu}, \boldsymbol{n} \rangle$. Clearly, for a given multiset $\mathcal{X}$ the half-space depth cannot exceed $\lfloor (\#\mathcal{X} - 1)/2 \rfloor$ where $\#$ symbolises cardinality but this value is not always realised.

As the last multivariate median concept, we mention the **convex-hull-stripping median** [3]. It is obtained by an iterative process: Starting with $\mathcal{X}_0 := \mathcal{X}$, one obtains $\mathcal{X}_{i+1}$ from $\mathcal{X}_i$, $i = 0, 1, 2, \ldots$, by removing all points $\boldsymbol{x}$ that lie on the boundary of the convex hull of $\mathcal{X}_i$. This is repeated until one finds i with $\mathcal{X}_i \neq \varnothing = \mathcal{X}_{i+1}$. Each point $\boldsymbol{\mu}$ in the convex hull of $\mathcal{X}_i$ then is a convex-hull-stripping median of the initial $\mathcal{X}$.

2.2 Equivariance

Equivariance essentially describes commutativity between some operator acting in a suitable space and transformations of this space. In the following we will distinguish whether the set of admissible transformations is independent of the actual data set or not, and speak of absolute or relative equivariance, respectively.

Basic definitions. Denote by $\mathcal{S}$ a suitable data space (e.g. $\mathbb{R}^d$). An operator $\boldsymbol{\varphi}$ mapping multisets $\mathcal{X}$ of values $\boldsymbol{x} \in \mathcal{S}$ to single values $\boldsymbol{\varphi}(\mathcal{X}) \in \mathcal{S}$ is called **absolutely equivariant** w.r.t. a set $\mathcal{T}$ of transformations $T : \mathcal{S} \to \mathcal{S}$ if for any multiset $\mathcal{X} \subset \mathcal{S}$ and any transformation $T \in \mathcal{T}$ one has $\boldsymbol{\varphi}(T\mathcal{X}) = T\boldsymbol{\varphi}(\mathcal{X})$. Note that $T\mathcal{X}$ here denotes simultaneous application of T to all elements of $\mathcal{X}$.

An operator φ as stated before is called **relatively equivariant** w.r.t. $\mathcal{T}$ if $\mathcal{T}$ is a set-valued operator that assigns to each multiset $\mathcal{X} \subset \mathcal{S}$ a set $\mathcal{T}(\mathcal{X})$ of transformations $T : \mathcal{S}' \to \mathcal{S}$ where $\mathcal{X} \cup \{\varphi(\mathcal{X}\} \subseteq \mathcal{S}' \subseteq \mathcal{S}$ such that for any $T \in \mathcal{T}(\mathcal{X})$ one has $\varphi(T\mathcal{X}) = T\varphi(\mathcal{X})$.

Equivariance is in fact a decisive feature when it comes to the application of filtering operators to given data. For example, application of an operator that possesses only Euclidean equivariance to data which do not have a meaningful Euclidean structure is dangerous as it implicitely imposes a random Euclidean structure on these data, and uses it to draw conclusions. This difficulty has in fact been a driving force behind the development of different multivariate median concepts in statistical literature.

The following equivariance properties of univariate and multivariate medians are largely known from the literature, see in particular [11] and the references therein.

Univariate median. The classic median possesses two strong equivariance properties that together form the essence of its outstanding role as a robust central position measure. First, it is equivariant under (the set of all) strictly monotonically increasing functions $T : \mathbb{R} \to \mathbb{R}$. This is the **morphological equivariance** mentioned in Section 1, which is obviously an absolute equivariance property. Second, there is the **radial scaling equivariance:** Given a finite multiset $\mathcal{X} \subset \mathbb{R}$ with median μ, the median is unchanged if each $x \in \mathcal{X}$ is replaced with some $\mu + c(x - \mu)$ where the factors $c > 0$ can even be chosen independently for each x. As the set of admissible transformations obviously depends on $\mathcal{X}$, namely, of its median, this is a relative equivariance. Finally, the univariate median is equivariant under reflections. Formally, this is also an absolute equivariance property which we will shortly refer to as **centrality.**

Centrality, understood as equivariance under reflections at arbitrary hyperplanes, is shared by all multivariate medians under discussion (intuitively, it is crucial for calling an operator a median, or more generally a mean). Regarding other equivariances, the multivariate medians vary, so we will shortly discuss each of them.

L^1 **median.** This median is much more restrictive in terms of absolute equivariance. It is equivariant under similarity transforms, i.e. under Euclidean transformations and global rescalings. On the other hand, it fully implements radial scaling equivariance as a relative equivariance property.

Oja and TR-L^1 medians. Both medians are absolutely equivariant under arbitrary affine transformations of the data space. Radial scaling equivariance holds for the Oja median in configurations where it is uniquely defined. Unfortunately, radial scaling equivariance is not preserved for the TR-L^1 median. We remark instead the following asymptotic property: Let a data multiset $\mathcal{X} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ be given. If radial rescaling weights c_i for the data $\boldsymbol{x}_i$ are chosen as $c_i = 1 + \varepsilon C_i$ with fixed C_i for the individual points and a global variable parameter ε, then for $\varepsilon \to 0$ the TR-L^1 median of the multiset $\mathcal{X}' = \{c_1\boldsymbol{x}_1, \ldots, c_N\boldsymbol{x}_N\}$ deviates from that of $\mathcal{X}$ by $\mathcal{O}(\varepsilon)$.

Half-space median. The half-space median shares with the previously mentioned two concepts the absolute equivariance under affine transformations. Moreover, as it depends only on the situation of points relative to straight lines, i.e. whether some point is located on the one or other side of that straight line, one can establish equivariance w.r.t. a somewhat larger set of global transformations, namely all projective transforms of the projective space $\mathrm{P}^d \supset \mathbb{R}^d$ that do not take any point from the convex hull of the data multiset $\mathcal{X}$ to infinity. As the set of admissible transformations depends on $\mathcal{X}$, this is a relative equivariance; we will refer to it as **restricted projective equivariance.** Note that for sequences $(\mathcal{X}_1, \mathcal{X}_2, \ldots)$ increasing beyond limits, i.e. with convex hulls $\mathrm{conv}(\mathcal{X}_i)$ that fulfil $\bigcup\limits_{i=1}^{\infty} \mathrm{conv}(\mathcal{X}_i) = \mathbb{R}^d$, the corresponding sets $\mathcal{T}_i$ of admissible projective transforms converge to the set $\mathcal{T}^*$ of affine transforms, $\mathcal{T}_1 \supset \mathcal{T}_2 \supset \ldots$ with $\bigcap_{i=1}^{\infty} \mathcal{T}_i = \mathcal{T}^*$ because affine transforms are the only projective transforms that take no finite point to infinity.

Radial scaling equivariance does in general not hold for the half-space median; however, it is valid for those data multisets $\mathcal{X}$ for which the half-space median attains the maximum possible half-space depth $\lfloor (\#\mathcal{X} - 1)/2 \rfloor$.

Convex-hull-stripping median. The equivariance properties of the convex-hull-stripping median resemble those of the half-space median as it possesses the same absolute affine equivariance and relative restricted projective equivariance. Radial scaling equivariance does not hold.

Generalisation of morphological equivariance. Looking back at the equivariance of the univariate median (and many morphological operators) under arbitrary monotone transformations of $\mathbb{R}$, the question arises what is the best counterpart one can establish for this in the multivariate case. For a tentative answer to this question, one can interpret increasing monotone transformations of $\mathbb{R}$ as orientation-preserving maps: they do not change the orientation of intervals, i.e. one-dimensional simplices $[x, y]$. Generalising this to the multivariate case, one is naturally led to consider transformations T of $\mathbb{R}^d$ that preserve the orientation of d-dimensional simplices $[\boldsymbol{x}_0, \ldots, \boldsymbol{x}_d]$. This boils down to requiring that the situation of any point in $\mathbb{R}^d$ relative to any hyperplane must not change. Postulating this for all points in $\mathbb{R}^d$, one obtains affine equivariance. Alternatively, restricting the requirement to the convex hull of a given data multiset, one obtains again the restricted projective equivariance.

We suggest therefore to consider restricted projective equivariance as multivariate morphological equivariance.

3 Space-Continuous Analysis

All definitions from Subsection 2.1 can be directly applied within median filtering procedures for discrete multivariate images. To study PDE limits, however, they need to be transferred to space-continuous multivariate images represented by smooth functions $\boldsymbol{u} : \mathbb{R}^2 \to \mathbb{R}^d$. The selection of values around a given location $\boldsymbol{x} \in \mathbb{R}^2$ is then accomplished using a compact neighbourhood of $\boldsymbol{x}$ as structuring element, and yields a density of intensities $\gamma : \mathbb{R}^d \to \mathbb{R}_0^+$ where $\mathbb{R}_0^+$ denotes the set of nonnegative real numbers. Given the smoothness of $\boldsymbol{u}$, γ has compact support and is absolutely integrable; it may be normalised to total weight 1.

Medians of multivariate densities. With the exception of the convex-hull-stripping median, the multivariate median concepts can easily be transferred to the case of compactly supported absolutely integrable densities γ, essentially by replacing sums with integrals, see [17]. The convex-hull-stripping median is more difficult to transfer; as shown in [18] the iterative process turns into a shape evolution process similar to the affine morphological scale space [2]. As an asymptotic analysis of the final point of this shape evolution has not been accomplished so far, we leave this median concept aside for the further discussion in this subsection.

Limiting process. Modelled after [4], we consider disc-shaped structuring elements $D_\varrho(\boldsymbol{x})$ of radius ϱ centered at $\boldsymbol{x}$ for the filtering of multivariate images $\boldsymbol{u}$ as specified above. The multivariate median of the density γ of image values within $D_\varrho(\boldsymbol{x})$ then is the value of the median-filtered image $\boldsymbol{M}_\varrho\boldsymbol{u}$. Similar to [4], one obtains results of the type $\lim\limits_{\varrho\to 0}\frac{\boldsymbol{M}_\varrho\boldsymbol{u}(\boldsymbol{x})-\boldsymbol{u}(\boldsymbol{x})}{\varrho^2/6} = L\boldsymbol{u}$ with some (spatial) differential operator L which justify to consider the time evolution PDE $\boldsymbol{u}_t = L\boldsymbol{u}$ as the asymptotic evolution for the respective multivariate median filter.

Equivariant normalisation. In [14, 16] asymptotic evolutions of multivariate median filters were derived. In doing so, it was helpful to exploit the Euclidean and affine equivariance, respectively, of the underlying median operators in the data space as well as the Euclidean equivariance in the image plane contributed by the structuring element D_ϱ to normalise the function $\boldsymbol{u}$ around the location $\boldsymbol{x}$.

In the bivariate case ($d = 2$) this is done as follows: First, translations in the image plane and data space are used to shift $\boldsymbol{x}$ and $\boldsymbol{u}(\boldsymbol{x})$ to $\boldsymbol{0}$. Next, rotations around $\boldsymbol{0}$ in the image plane and data space are applied to make the Jacobian $\boldsymbol{Du}(\boldsymbol{0})$ diagonal and positive semidefinite (at generic locations: positive definite). We call the normalisation up to this step **Euclidean normalisation.** Furthermore, if the median under consideration admits affine equivariance, an affine transform in the data space can be used to rescale the data (at non-degenerate locations) such that the Jacobian becomes the unit matrix. We refer to this as **affine normalisation.** Note that this is the continuous counterpart of the normalisation by the covariance matrix in the definition of the TR-L^1 median.

In the case $d > 2$, essentially the same kind of normalisation can be applied; however, the Jacobian is now a $d \times 2$-matrix and will be transformed in a way that its third and further rows are zero, and the 2×2-submatrix consisting of the first two rows satisfies the requirements (diagonal, positive semidefinite, positive definite, unit matrix) as specified before.

In the following we state the approximation results from [14, 16] for the normalised cases; the general equations are obtained from these by applying the respective inverse transforms to the PDE $\boldsymbol{u}_t = L\boldsymbol{u}$. For brevity we focus on the bivariate case ($d = 2$).

Normalised PDE approximations of multivariate median filtering. In [14] it was shown that bivariate L^1 median filtering of an image $\boldsymbol{u} : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y)^\mathrm{T} \mapsto (u, v)^\mathrm{T}$ in Euclidean normalisation approximates the PDE system

$$\begin{aligned} u_t &= Q(u_x/v_y)u_{xx} + (1 - Q(u_x/v_y))u_{yy} - 2(u_x/v_y)Q(u_x/v_y)v_{xy} \\ v_t &= (1 - Q(v_y/u_x))v_{xx} + Q(v_y/u_x)v_{yy} - 2(v_y/u_x)Q(v_y/u_x)u_{xy} \end{aligned} \tag{1}$$

with a coefficient function Q that can be stated in terms of elliptic integrals.

Specialising to the affine normalised situation, one has $Q(1) = 1/4$, thus

$$u_t = \tfrac{1}{4}u_{xx} + \tfrac{3}{4}u_{yy} - \tfrac{1}{2}v_{xy}\ , \qquad v_t = \tfrac{3}{4}v_{xx} + \tfrac{1}{4}v_{yy} - \tfrac{1}{2}u_{xy}\ , \tag{2}$$

which is the PDE system for the Oja and TR-L^1 median filters in affine normalisation. In [16] it was proven that bivariate half-space median filtering approximates the same PDE system. For the L^1, Oja and TR-L^1 median filters also trivariate versions of these PDE systems are found in [14].

Equivariance. The definitions of absolute and relative equivariance translate straightforward to the case of PDEs evolutions. As can be expected, the PDE evolutions for L^1 median filtering such as (1) are equivariant under similarity transformations of the data space; the PDE evolutions such as (2) for the other medians are affine equivariant. Moreover, the fact that (2) also corresponds to the half-space median lets expect restricted projective equivariance which indeed holds. Remarkably, the just affine equivariance of the Oja and TR-L^1 medians is upgraded to restricted projective equivariance in the asymptotic limit.

4 Derivation of PDE Evolutions by Equivariance

In this section we turn around to derive bivariate image filtering PDEs from equivariance properties modelled after median filters. We start by assuming that $\boldsymbol{u}$ is a smooth bivariate image evolution which is described by some PDE system $\boldsymbol{u}_t = L\boldsymbol{u}$. To restrict the PDE system, we impose conditions one by one, modelled after the properties of median operators.

(I) Translation equivariance. This allows us to shift the location of interest to $\mathbf{0}$ with $\boldsymbol{u}(\mathbf{0}) = \mathbf{0}$. We write down the spatial Taylor expansion up to second order at a non-degenerate location $\mathbf{0}$. Suppressing for the moment the time parameter, and considering only $\boldsymbol{x} = (x,y)^{\mathrm{T}} \in D_\varrho(\mathbf{0})$, we have

$$\begin{aligned} u(x,y) &= \alpha_1 x + \alpha_2 y + \tfrac{1}{2}\beta x^2 + \gamma xy + \tfrac{1}{2}\delta y^2 + \mathcal{O}(\varrho^3)\;, \\ v(x,y) &= \alpha_1' x + \alpha_2' y + \tfrac{1}{2}\beta' x^2 + \gamma' xy + \tfrac{1}{2}\delta' y^2 + \mathcal{O}(\varrho^3) \end{aligned} \tag{3}$$

where we have replaced first and second order derivatives of $\boldsymbol{u}$ at $\mathbf{0}$ with variables.

Noting that medians of $\boldsymbol{u}$ within D_ϱ are $\mathcal{O}(\varrho^2)$, we seek a PDE evolution that is approximated by some filtering process in the limit $\varrho \to 0$ with step size $\mathcal{O}(\varrho^2)$. This implies that the PDE evolution is described by a bivariate function $\boldsymbol{p} = (p,q)^{\mathrm{T}}$ of the first and second order derivatives of $\boldsymbol{u}$ as

$$\boldsymbol{u}_t = \boldsymbol{p}(\alpha_1, \alpha_2, \beta, \gamma, \delta, \alpha_1', \alpha_2', \beta', \gamma', \delta')\;. \tag{4}$$

(II) Centrality. We impose first centrality in its weakest form, w.r.t. the reflection on the origin, which implies

$$\boldsymbol{p}(\alpha_1, \alpha_2, 0, 0, 0, \alpha_1', \alpha_2', 0, 0, 0) = \mathbf{0}\;. \tag{5}$$

(III) Scaling equivariance. With this requirement it follows that $\boldsymbol{p}$ is homogeneous of degree 0 in α_1, α_2, α_1' and α_2' and of degree 1 in the remaining parameters,

$$\boldsymbol{p}(\lambda\alpha_1, \lambda\alpha_2, \mu\beta, \mu\gamma, \mu\delta, \lambda\alpha_1', \lambda\alpha_2', \mu\beta', \mu\gamma', \mu\delta') = \mu\boldsymbol{p}(\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta') \tag{6}$$

for $\lambda > 0$, $\mu > 0$.

(IV) Euclidean equivariance. Now we can apply Euclidean normalisation. In the normalised setting, we have $\alpha_2 = \alpha'_1 = 0$. By homogeneity, $\boldsymbol{p}$ in fact only depends on the ratio α_1/α'_2 instead of the two individual variables.

(V) Affine equivariance. By affine normalisation we achieve $\alpha'_1 = \alpha_2 = 1$, thus only the second order derivatives are left as parameters for $\boldsymbol{p} = \boldsymbol{p}(\beta, \gamma, \delta, \beta', \gamma', \delta')$.

We notice that in the affine normalised setting, there is a further degree of freedom: Simultaneous rotations and reflections in the image (x-y) and data (u-v) plane leave $\alpha_1 = \alpha'_2 = 1$ untouched but transform the second order derivatives. Thus, $\boldsymbol{p}$ must be equivariant under these operations.

In particular, reflections on the y and v axes and similarly on the x and u axes imply

$$p(0, \gamma, 0, \beta', 0, \delta') = 0 \;, \qquad\qquad q(\beta, 0, \delta, 0, \gamma', 0) = 0 \tag{7}$$

as well as $\boldsymbol{p}(-\beta, -\gamma, -\delta, -\beta', -\gamma', -\delta') = -\boldsymbol{p}(\beta, \gamma, \delta, \beta', \gamma', \delta')$. By reflection on the diagonal $x = y$, we find $q(\beta, \gamma, \delta, \beta', \gamma', \delta') = p(\delta', \gamma', \beta', \delta, \gamma, \beta)$, thus reducing the problem to finding a single univariate function p. Using general rotations with rotation matrix $\boldsymbol{R} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$ simultaneously in the x-y and u-v planes and taking first-order derivatives w.r.t. the rotation angle φ yields the differential equations

$$p_\beta - p_\delta - p_{\gamma'} = 0 \;, \qquad\qquad p_{\beta'} - p_{\delta'} - p_\gamma = 0 \tag{8}$$

for p. Furthermore, second derivatives w.r.t. φ yield the additional conditions $p_\gamma = p_{\beta'} = p_{\delta'} = 0$, from which together with (7) we see that p is a 1-homogeneous function of only β, δ and γ'. According to Euler's Homogeneous Function Theorem p can be represented in the form

$$p(\beta, \delta, \gamma') = \beta p_\beta + \delta p_\delta + \gamma' p_{\gamma'} \;. \tag{9}$$

Simplifying (9) with (8) we obtain the following intermediate result.

Proposition 1. *A bivariate PDE evolution in affine normalisation which is associated to a local filtering operator with centrality property and affine equivariance can only be of the form*

$$u_t = p(u_{xx}, u_{yy}, v_{xy}) \;, \qquad\qquad v_t = p(v_{yy}, v_{xx}, u_{xy}) \tag{10}$$

with a 1*-homogeneous function* p *that satisfies*

$$p(\beta, \delta, \gamma') = (\beta + \gamma')p_\beta + (\delta - \gamma')p_\delta \;. \tag{11}$$

(VI) Relative equivariances. For further specification we need an additional requirement that can be derived from several relative equivariances. If we assume radial scaling equivariance, we can in particular replace (in the normalised setting under consideration) $\boldsymbol{u}(\boldsymbol{x})$ for each $\boldsymbol{x} \in D_\varrho$ with the scalar multiple $(1 + \varepsilon\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x})$ for some small ε and require that $\boldsymbol{u}_t$ remains unchanged. This implies

$$p(\beta + 2\varepsilon, \delta, \gamma' + \varepsilon) = p(\beta, \delta, \gamma') \;. \tag{12}$$

Unfortunately, as discussed earlier, radial scaling equivariance does not hold for all multivariate median concepts in our investigation. Among the affine equivariant medians, it holds only for the Oja median (where the restriction to unique cases is no problem in the space-continuous case at generic locations). However, (12) can be derived alternatively from restricted projective equivariance (as it holds for the half-space median). Moreover, it can be shown that also the asymptotic radial scaling equivariance which holds for the TR-L^1 median is sufficient to ensure (12) since the effect of the radial rescaling with $(1+\varepsilon \boldsymbol{x})$ for $\boldsymbol{x} \in D_\varrho$ on $\boldsymbol{p}$ is of order $\mathcal{O}(\varrho\epsilon)$ and thereby vanishes in the limit $\varrho \to 0$.

Inserting (11) into (12) we obtain $(\beta+\gamma'+3\varepsilon)p_\beta + (\delta-\gamma'-\varepsilon)p_\delta = (\beta+\gamma')p_\beta + (\delta-\gamma')p_\delta$ and finally

$$p_\delta = 3p_\beta \ . \tag{13}$$

Together with (11) this yields $p(\beta,\delta,\gamma') = (\beta+3\delta-2\gamma')p_\beta$ which implies that p_β is constant and p a linear function. The single degree of freedom is the choice of p_β which amounts to a time rescaling. The consequence is our second result, summarised in the following proposition.

Proposition 2. *A bivariate PDE evolution in affine normalisation as in Prop. 1 which additionally satisfies asymptotic radial scaling equivariance or restricted projective equivariance is necessarily of the form* (2).

We remark that Prop. 2 explains the coincidence of the PDE asymptotics of Oja, TR-L^1 and half-space median in the bivariate case. Generalisations on one hand to trivariate and generally multivariate evolutions, and on the other hand to Euclidean equivariance are part of ongoing work.

The result of Prop. 1 states a more general class of affine equivariant PDE evolutions that are in a sense close to median-associated ones but without the last requirement of relative equivariances. Functions p that satisfy (11) for real β, δ, γ' with $\beta+\gamma' > 0$ and $\delta-\gamma' > 0$ are e.g. given by

$$p(\beta,\delta,\gamma') = \bigl((\beta+\gamma')^s + \vartheta(\delta-\gamma')^s\bigr)^{1/s} \tag{14}$$

with arbitrary parameters $\vartheta > 0$, $s > 0$, which includes the linear case for $s = 1$. To be usable for the PDE image filter, however, p needs to be defined on the entire parameter space $(\beta,\delta,\gamma') \in \mathbb{R}^3$. Such an extension is obviously possible for some values of s, particularly $s = m$ or $s = 1/m$ for odd natural numbers m. We believe that this larger class of PDE evolution deserves further study, and include some numerical examples in the next section.

5 Experiments

While the emphasis of this paper is largely on theory, we want to give an impression of the effect of the filters under consideration by a numerical example. Although practical relevance is expected rather for multivariate images with at least three channels such as RGB colour images or diffusion tensor images, it appears appropriate to stay in the

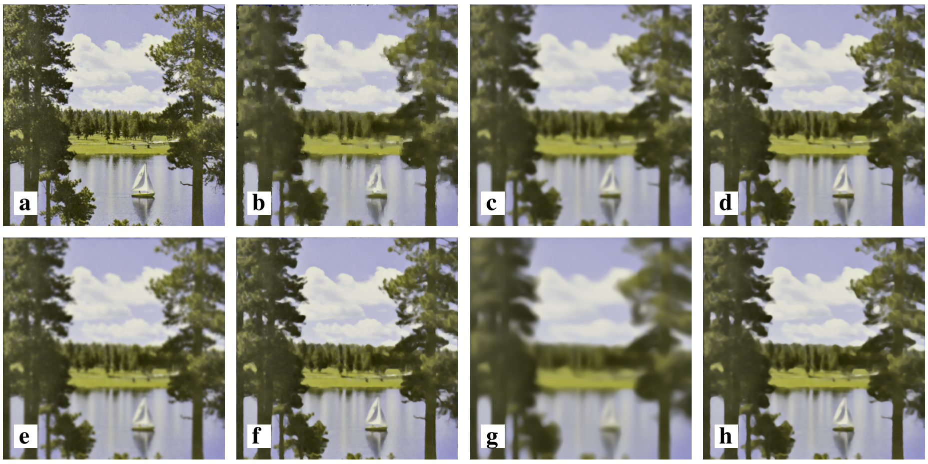

Fig. 1. Filtering of a bivariate colour image (512×512 pixels). **a** Colour image *sailboat* reduced to yellow-blue colour space. **b** Result of half-space median filtering with a discrete disc of radius 2 as structuring element, 15 iterations. **c** Corresponding evolution by the PDE system (10), (14), $s = 1$, $\vartheta = 3$, up to time $T = 2.5$ (60 time steps of size 0.041665); same evolution as (2) except for speed-up by a factor 4. **d** Same as c but with heuristic anti-diffusion to reduce numerical dissipation; see text. **e** PDE evolution (10), (14), $s = 3$, $\vartheta = 3$, $T = 2.5$ (same time steps as c, d). **f** Same as e but with heuristic anti-diffusion. **g** PDE evolution (10), (14), $s = 1/3$, $\vartheta = 3$, $T = 0.8333$ (1000 time steps of size 0.0008333). **h** Same as g but with heuristic anti-diffusion.

bivariate setting in accordance with the analysis presented. As an example of a bivariate image we therefore present a colour image where the RGB colour space has been reduced to a yellow-blue (YB) colour space by averaging the red and green channels. All algorithms were implemented in C++.

Numerical aspects. Whereas the PDE system (10), (14) is stated in affine normalisation, practical computation by a finite-difference scheme is best done by applying only Euclidean normalisation to a 3×3 patch and evaluating the PDE system therefore in the form

$$u_t = p(u_{xx}, u_{yy}, v_{xy}/v_y) \;, \qquad v_t = p(v_{yy}, v_{xx}, u_{xy}/u_x) \;. \tag{15}$$

Still, as already noted in [14], a straightforward discretisation by central differences is unstable. In [14, App. 6] a stable numerical scheme was devised that uses in particular min-mod stabilised upwind discretisations for the terms involving u_{xy}/u_x, v_{xy}/v_y. We use this scheme with minor adaptations to suit the more general function p from (14).

As finite difference discretisations of PDEs tend to add undesired blur to the results, it was proposed in [14] to modify the coefficients of the PDE evolutions by an anti-diffusion term which can be safely done just by reducing the coefficients of u_{xx}, u_{yy}, v_{xx}, v_{yy} by a uniform amount. We follow this recommendation and include exemplary results with this compensation.

Exemplary results. Based on the original image shown in Fig. 1a, we show first a result of iterated half-space median filtering, Fig. 1b, using the implementation from [16]. Frame c shows the result of the corresponding PDE system (10), (14) with $s = 1$ at the appropriate evolution time, whereas frame d represents the same with the anti-diffusion compensation. Frames e–f show visually similar results obtained with $s = 3$ instead of $s = 1$. In frames g–h we used $s = 1/3$. To achieve a visually comparable degree of image smoothing, adjustments were required both for the total evolution time (reduced by a factor of 3) but also for the time step size (reduced by a factor of 50, necessitating dramatically more iterations).

Still, the computation time for all of the PDE evolutions is much less than that for the half-space median computation. With our C++ implementations that were in no way optimised for performance, computation times for single-core computation on an AMD Phenom(tm) II X6 1100T processor (manufactured around 2011) running at 3.2 GHz under Ubuntu Linux 20.04 ranged from seconds to minutes for the PDE evolutions whereas more than two hours were necessary for the half-space median filtering. We believe that significant speedups are possible by more efficient implementations.

6 Summary and Conclusions

In this paper we have re-visited previous results on multivariate image filtering PDE systems associated with median filters with special emphasis on their equivariance properties. We have presented a systematic derivation of such PDE systems on the basis of equivariance properties, for the time being in the bivariate case. As a result, we showed that the approximation of the same PDE system by several affine equivariant median filtering processes is no coincidence, but a necessity. As a by-product we have identified a more general class of PDE evolutions with homogeneous functions of second derivatives as right-hand sides that appear to be worth further study.

Ongoing work is directed at generalising the result of this paper to the general multivariate situation including practically meaningful cases like RGB colour images and diffusion tensor images. We also aim at extending the analysis to the case of Euclidean equivariance where part of our present line of argument cannot be transferred straightforwardly. As mentioned before, the larger class of homogeneous evolutions described above is of interest for further investigation, too.

Another direction for future research is the adequate interpretation of the PDE systems in question. The clear geometric intuition of curvature flow in the univariate case is so far not reflected in an appropriate understanding of the multivariate PDE evolution. A geometric interpretation will definitely strengthen the theoretical framework and promote applicability.

References

1. Alvarez, L., Guichard, F., Lions, P.L., Morel, J.M.: Axioms and fundamental equations in image processing. Archive for Rational Mechanics and Analysis **123**, 199–257 (1993)
2. Alvarez, L., Morales, F.: Affine morphological multiscale analysis of corners and multiple junctions. International Journal of Computer Vision **25**, 95–107 (1994)

3. Green, P.J.: Peeling bivariate data. In: Barnett, V. (ed.) Interpreting multivariate data, pp. 3–20. Wiley, Chichester (1981)
4. Guichard, F., Morel, J.M.: Partial differential equations and image iterative filtering. In: Duff, I.S., Watson, G.A. (eds.) The State of the Art in Numerical Analysis, pp. 525–562. No. 63 in IMA Conference Series (New Series), Clarendon Press, Oxford (1997)
5. Hettmansperger, T.P., Randles, R.H.: A practical affine equivariant multivariate median. Biometrika **89**(4), 851–860 (2002)
6. Kleefeld, A., Breuß, M., Welk, M., Burgeth, B.: Adaptive filters for color images: median filtering and its extensions. In: Trémeau, A., Schettini, R., Tominaga, S. (eds.) Computational Color Imaging, Lecture Notes in Computer Science, vol. 9016, pp. 149–158. Springer, Cham (2015)
7. Oja, H.: Descriptive statistics for multivariate distributions. Statistics and Probability Letters **1**, 327–332 (1983)
8. Rao, C.R.: Methodology based on the L_1-norm in statistical inference. Sankhyā A **50**, 289–313 (1988)
9. Sethian, J.A.: Level Set Methods: Evolving Interfaces in Geometry, Fluid Mechanics, Computer Vision and Material Sciences. Cambridge University Press, Cambridge, UK (1996)
10. Small, C.G.: Measures of centrality for multivariate and directional distributions. Canadian Journal of Statistics **15**(1), 31–39 (1987)
11. Small, C.G.: A survey of multidimensional medians. International Statistical Review **58**(3), 263–277 (1990)
12. Spence, C., Fancourt, C.: An iterative method for vector median filtering. In: Proc. 2007 IEEE International Conference on Image Processing. vol. 5, pp. 265–268 (2007)
13. Weber, A.: Über den Standort der Industrien. Mohr, Tübingen (1909)
14. Welk, M.: Multivariate median filters and partial differential equations. Journal of Mathematical Imaging and Vision **56**, 320–351 (2016)
15. Welk, M.: PDE for bivariate amoeba median filtering. In: Angulo, J., Velasco-Forero, S., Meyer, F. (eds.) Mathematical Morphology and its Applications to Signal and Image Processing, Lecture Notes in Computer Science, vol. 10225, pp. 271–283. Springer, Cham (2017)
16. Welk, M.: Asymptotic analysis of bivariate half-space median filtering. In: Roth, P.M., Steinbauer, G., Fraundorfer, F., Brandstötter, M., Perko, R. (eds.) Proceedings of the Joint Austrian Computer Vision and Robotics Workshop, pp. 151–156. Verlag der Technischen Universität Graz, Graz (2020). https://doi.org/10.3217/978-3-85125-752-6-34
17. Welk, M.: Multivariate medians for image and shape analysis. Tech. Rep. eess.IV:1911.00143v2, arXiv.org (2021), http://arxiv.org/abs/1911.00143
18. Welk, M., Breuß, M.: The convex-hull-stripping median approximates affine curvature motion. In: Burger, M., Lellmann, J., Modersitzki, J. (eds.) Scale Space and Variational Methods in Computer Vision, Lecture Notes in Computer Science, vol. 11603, pp. 199–210. Springer, Cham (2019)
19. Welk, M., Feddern, C., Burgeth, B., Weickert, J.: Median filtering of tensor-valued images. In: Michaelis, B., Krell, G. (eds.) Pattern Recognition, Lecture Notes in Computer Science, vol. 2781, pp. 17–24. Springer, Berlin (2003)
20. Welk, M., Kleefeld, A., Breuß, M.: Non-adaptive and amoeba quantile filters for colour images. In: Benediktsson, J.A., Chanussot, J., Najman, L., Talbot, H. (eds.) Mathematical Morphology and its Applications to Signal and Image Processing, Lecture Notes in Computer Science, vol. 9082, pp. 398–409. Springer, Cham (2015)
21. Welk, M., Weickert, J., Becker, F., Schnörr, C., Feddern, C., Burgeth, B.: Median and related local filters for tensor-valued images. Signal Processing **87**, 291–308 (2007)