PDE for Bivariate Amoeba Median Filtering

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Abstract. Amoebas are image-adaptive structuring elements for morphological filters that have been introduced by Lerallut et al. in 2005. Iterated amoeba median filtering on grey-scale images has been proven to approximate asymptotically for vanishing structuring element radius a partial differential equation (PDE) which is known in image filtering by the name of self-snakes. This approximation property helps to understand the properties of both, morphological and PDE, image filter classes. Recently, also the PDEs approximated by multivariate median filtering with non-adaptive structuring elements have been studied. Affine equivariant multivariate medians turned out to yield more favourable PDEs than the more popular L^1 median. We continue this work by considering amoeba median filtering of bivariate images using affine equivariant medians. We prove a PDE approximation result for this case. We validate the result by numerical experiments on example functions sampled with high spatial resolution.

1 Introduction

Image processing methods based on superficially disparate paradigms often show surprising similarities in their results. For example, discrete image filters designed in a morphological frameworks can often be connected to partial differential equations (PDEs). Dilation and erosion can be linked to Hamilton–Jacobi PDEs [2]; the median filter is known to approximate in a continuous limit a curvature motion PDE [10]. The important role of adaptive filters in all branches of image processing has triggered interest in extending such connections also to adaptive filters.

In mathematical morphology, adaptive filters can be based on adaptive structuring elements. Morphological amoebas which have been introduced by Lerallut, Decencière and Meyer [14, 15] are one class of such image-adaptive structuring elements. The essence of the amoeba construction is the definition of a spatially variant metric on the image domain that combines the spatial distance with local image contrast. The structuring element of a pixel is then defined as a neighbourhood of prescribed radius with regard to this metric. Thereby, preferably pixels with similar intensities are included. The shape of the structuring element then adapts flexibly to image details, giving reason to the name amoebas.

Morphological amoebas are naturally linked to graph morphology [21]. They have been compared to further types of adaptive structuring elements [6, 9, 22]. They have also been employed to construct an active contour method for image segmentation [25].

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Amoeba-based filters for multi-channel images, such as colour images or diffusion tensor data sets, have also been considered [8, 13].

For univariate amoeba median filtering, it has been shown [29] that it approximates the so-called *self-snakes* PDE [18] in a space-continuous limit. In this paper, we want to investigate amoeba median filtering of multi-channel images under this aspect. In recent image processing literature, multivariate median filtering is mostly based on the so-called L^1 median [13, 20, 30]. For multivariate median filtering using the L^1 median, a PDE limit becomes fairly complicated already in the non-adaptive case [28]. However, in [27] it was shown that two affine equivariant median concepts, the Oja median [16] and the transformation–retransformation L^1 median [5, 12, 17], give rise to image filters that can be related to simpler and more manageable PDEs.

Our contribution. Motivated by the previously mentioned results, we study amoeba filters based on the affine equivariant Oja and transformation–retransformation L^1 median. For the purpose of the present paper, we restrict ourselves to the bivariate case, i.e. two-channel images, and one specific amoeba metric. We derive a PDE that is approximated by the amoeba median filter under consideration. More precisely, one step of amoeba median filtering in a space-continuous setting asymptotically approximates a time step of an explicit scheme for the PDE when the radius of structuring elements tends to zero; the time step size goes to zero with the square of the radius.

The focus of our contribution is theoretic. Therefore we refrain from presenting image filtering experiments; apart from flow fields, there are not many meaningful application cases for bivariate images. Instead we will validate our approximation results by numerical experiments on example functions sampled with high spatial resolution.

Structure of the paper. We recall the main facts on amoeba filtering procedures in Section 2. Section 3 is devoted to multivariate median concepts. Our analysis of bivariate amoeba median filtering is presented in Section 4, followed by the numerical validation in Section 5. A conclusion is given in Section 6.

2 Amoeba Filtering

Discrete amoeba filtering. The construction of amoeba structuring elements [14] relies on an *amoeba metric.* Given a discrete image $\boldsymbol{u} = (u_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathcal{J}}$ over a discrete domain such as $\mathcal{J} = \{1,\dots,M\} \times \{1,\dots,N\}$, the amoeba metric measures the distance between two neighbouring pixels $\boldsymbol{i},\boldsymbol{j} \in \mathcal{J}$ as $d(\boldsymbol{i},\boldsymbol{j}) := \varphi\big(\|\boldsymbol{i}-\boldsymbol{j}\|, \beta\,|u_{\boldsymbol{i}} - u_{\boldsymbol{j}}|\big)$, where $\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a 1-homogeneous continuous function. In [14] the L^1 sum $\varphi(s,t) \equiv \varphi_1(s,t) := s + t$ is used; an alternative is the L^2 sum $\varphi(s,t) \equiv \varphi_2(s,t) := \sqrt{s^2 + t^2}$. The neighbourhood relation may be defined by 4-neighbourhoods as in [14] or 8-neighbourhoods; other choices are possible. The contrast-scale parameter $\beta > 0$ balances the influence of spatial and intensity information.

Summation along paths $P := (\boldsymbol{i}_0, \boldsymbol{i}_1, \dots, \boldsymbol{i}_k)$ where $\boldsymbol{i}_j$ and $\boldsymbol{i}_{j+1}$ for $j = 0,\dots,k-1$ are neighbours yields the path length $L(P) = \sum_{j=0}^{k-1} d(\boldsymbol{i}_j, \boldsymbol{i}_{j+1})$. For each pixel $\boldsymbol{i} \in \mathcal{J}$ one then defines the amoeba structuring element of radius ϱ as the set of all pixels $\boldsymbol{j} \in \mathcal{J}$ for which a path P from $\boldsymbol{i}$ to $\boldsymbol{j}$ exists that has $L(P) \le \varrho$.

Amoebas can be used in a straightforward way as structuring elements for e.g. dilation, erosion, or median filtering.

Continuous amoeba filtering. [29] This procedure can easily be translated to a continuous-scale setting: Assume $u : \mathbb{R}^2 \supset \Omega \to \mathbb{R}$ is a smooth function over the compact image domain Ω. The construction is understood best by considering the (rescaled) *image graph* $\Gamma := \{(\boldsymbol{x}, \beta\, u(\boldsymbol{x})) \mid \boldsymbol{x} \in \Omega\} \subset \Omega \times \mathbb{R}$. Note that Γ is a section in the bundle $\Omega \times \mathbb{R}$. A continuous amoeba metric on Ω can then be defined by introducing the infinitesimal metric $\mathrm{d}s := \varphi\big(\|\mathrm{d}\boldsymbol{x}\|, \beta\,|\mathrm{d}u|\big)$ on Γ, with $\|\cdot\|$ denoting some norm on $\mathbb{R}^2$, and φ as above, and projecting it back to Ω. In general $\mathrm{d}s$ will be a Finsler metric. For the special choice where $\|\cdot\|$ is the Euclidean norm, and $\varphi \equiv \varphi_2$, the metric on Γ is induced from the Euclidean metric on $\Omega \times \mathbb{R}$, yielding a Riemannian metric $\mathrm{d}s$ on Ω. In any case, it can be integrated along continuous curves in Ω, and gives rise to distances $d(\boldsymbol{x}, \boldsymbol{y})$ between points $\boldsymbol{x}, \boldsymbol{y} \in \Omega$. The continuous-scale amoeba $\mathcal{A}_\varrho(\boldsymbol{x})$ of radius ϱ around $\boldsymbol{x} \in \Omega$ is then the set of all $\boldsymbol{y} \in \Omega$ for which $d(\boldsymbol{x}, \boldsymbol{y}) \le \varrho$ holds.

Again, it is straightforward to conceive a continuous-scale amoeba filter: For each location $\boldsymbol{x} \in \Omega$, one selects the neighbourhood $\mathcal{A}_\varrho(\boldsymbol{x})$. Using the Borel measure on Ω, the intensities $u(\boldsymbol{y})$ for $\boldsymbol{y} \in \mathcal{A}_\varrho(\boldsymbol{x})$ give rise to a density on (a subset of) $\mathbb{R}$. The filtered value $v(\boldsymbol{x})$ is then obtained by applying an aggregation operator such as maximum (for dilation), minimum (for erosion), or median to the density. For example, the median of the density is the infimum μ of values $z \in \mathbb{R}$ for which $u(\boldsymbol{y}) \le z$ on at most half of the area of $\mathcal{A}_\varrho(\boldsymbol{x})$; i.e. in generic cases the level line $u(\boldsymbol{y}) = \mu$ bisects the amoeba area.

Continuous amoeba filtering of multivariate images. [13] The amoeba construction can easily be transferred to images which take values in $\mathcal{R} = \mathbb{R}^m$, such as RGB colour images ($m = 3$), planar flow fields ($m = 2$), or symmetric 2×2 matrices ($m = 3$). Let a continuous-scale multivariate image $\boldsymbol{u} : \mathbb{R}^2 \supset \Omega \to \mathbb{R}^m$ be given. Its graph $\Gamma := \{(\boldsymbol{x}, \beta\, \boldsymbol{u}(\boldsymbol{x})) \mid \boldsymbol{x} \in \Omega\}$ is a section in the bundle $\Omega \times \mathbb{R}^m$. Choosing a norm $\|\cdot\|_{\mathcal{R}}$ on $\mathbb{R}^m$ one defines $\mathrm{d}s := \varphi\big(\|\mathrm{d}\boldsymbol{x}\|, \beta\,\|\mathrm{d}\boldsymbol{u}\|_{\mathcal{R}}\big)$ on Γ and obtains the amoeba metric by projection to Ω. The construction of amoebas $\mathcal{A}_\varrho(\boldsymbol{x})$ then translates verbatim.

Multivariate amoeba filters can then be defined in the same way as grey-value filters, provided suitable aggregation operators are available on the density of values $\boldsymbol{u}(\boldsymbol{y}) \in \mathbb{R}^m$ for $\boldsymbol{y} \in \mathcal{A}_\varrho(\boldsymbol{x})$. Examples of such operators are supremum/infimum operators for symmetric matrices [4, 3], multivariate medians [11, 13, 16, 20, 30] or quantiles [31]. We will discuss multivariate median concepts in more detail in the next section.

Specification of amoeba metric for this paper. We will focus on continuous-scale amoeba filtering of bivariate images $u : \Omega \to \mathbb{R}^2$, where the amoeba metric is induced from the Euclidean metric in $\Omega \times \mathbb{R}^2 \subset \mathbb{R}^4$, i.e. $\|\cdot\|_{\mathcal{R}}$ is the Euclidean norm, and $\varphi \equiv \varphi_2$. From now on, we will refer to this setting as *Euclidean amoeba metric*.

3 Multi-Channel Median

Attempts to generalise the concept of median from the univariate setting to multivariate data go back more than a century, see the work by Hayford from 1902 [11].

L^1 median. One of the most popular multivariate median concepts, which is nowadays termed L^1 median, has been introduced as early as 1909 by Weber to location theory [23] and by Gini in 1929 to statistics [7]; it reappeared later in [1, 7, 24] and many other works. The L^1 median generalises the well-known property of the univariate median to minimise the sum of distances to given data values on the real line; thereby, the L^1 median of points $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{R}^m$ is defined as

$$\boldsymbol{\mu}_{L^1}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) := \operatorname*{argmin}_{\boldsymbol{x} \in \mathbb{R}^m} \sum_{i=1}^{n} \|\boldsymbol{x} - \boldsymbol{a}_i\| \tag{1}$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^m$. The L^1 median concept has been used in image processing for symmetric matrices [30] as well as for RGB images [13, 20]. A first attempt to derive the PDE approximated by such a median filter with non-adaptive structuring elements has been made in [28], with a later correction in [27].

Affine equivariance. The L^1 median is equivariant under similarity transformations of $\mathbb{R}^m$ (combinations of Euclidean transformations and scalings): Let T be such a transform; then we have $\boldsymbol{\mu}_{L^1}(T(\boldsymbol{a}_1), \ldots, T(\boldsymbol{a}_n)) = T(\boldsymbol{\mu}_{L^1}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n))$. However, the univariate median is equivariant under much more general transformations, namely under arbitrary strictly monotonous mappings of $\mathbb{R}$. Given that a Euclidean structure is often unnatural to impose on data sets, interest in alternative multivariate median concepts that feature at least affine equivariance has arised. From the various concepts developed for this purpose, see [19] for an overview, we mention two approaches.

Oja median. The first concept was introduced in 1983 by Oja [16] and termed *simplex median;* nowadays also the name *Oja median* has gained popularity. It generalises the same property of the univariate median as mentioned above. However, instead of viewing $|x - a_i|$ on $\mathbb{R}$ as a *distance,* it views it as *measure of an interval* on the real line, thus a one-dimensional simplex. Replacing one-dimensional simplices with m-dimensional ones, the simplex median is of points $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{R}^m$ is defined as

$$\boldsymbol{\mu}_{\mathrm{Oja}}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) := \operatorname*{argmin}_{\boldsymbol{x} \in \mathbb{R}^m} \sum_{1 \le i_1 < \ldots < i_m \le n} |[\boldsymbol{x}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m]| \tag{2}$$

where $|[\boldsymbol{x}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m]|$ is the m-dimensional volume of the simplex spanned by the $m + 1$ points $\boldsymbol{x}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$.

Transformation–retransformation L^1 median. An alternative approach is to upgrade the L^1 median to affine equivariance by a transformation–retransformation procedure [5, 12, 17]. Here, an affine transformation is determined based on the covariance matrix of input data, and the data thereby transformed to a normalised form with an isotropic covariance matrix. Then the L^1 median is computed and transformed back to the original coordinates. This approach allows to combine the more favourable algorithmic complexity of the L^1 median, as compared to the Oja median, with affine equivariance.

In the context of iterated median filtering of multivariate images, the Oja median has been studied in [26] for bivariate images; in [27] both Oja and transformation–retransformation L^1 median were considered in bivariate and trivariate settings, and

PDEs approximated by both types of median filters with non-adaptive structuring elements were derived. Compared to the PDE mentioned earlier for the standard L^1 median filter, these PDEs are more favourable as their coefficient functions are simpler, and coincide for both filter types. This means that, despite the fact that Oja and transformation–retransformation L^1 median of point sets are not the same, their corresponding image filters can be viewed as different discrete realisations of the same affine equivariant median filter concept [27].

Median concepts studied in this paper. Based on the encouraging findings in [27], we focus here on affine equivariant median filters for bivariate images based on the Oja and transformation–retransformation L^1 median.

4 PDE for Affine Equivariant Bivariate Amoeba Median Filtering

To begin with amoeba median filtering with Oja median, we can state our first result.

Theorem 1. *Let a bivariate image* $\boldsymbol{u} : \mathbb{R}^2 \supset \Omega \to \mathbb{R}^2$, $(x, y) \mapsto (u, v)$ *be given, for which the Jacobian* $\mathrm{D}\boldsymbol{u}$ *of* $\boldsymbol{u}$ *has rank* 2 *throughout* Ω. *For fixed contrast scale* $\beta > 0$, *and amoeba radius* $\varrho \to 0$, *one step of amoeba median filtering of* $\boldsymbol{u}$ *with Oja median approximates a time step of size* $\tau = \varrho^2/24$ *of an explicit scheme for the PDE system*

$$\partial_t \boldsymbol{u} = \boldsymbol{T}_1(\mathrm{D}\boldsymbol{u})\partial_{\boldsymbol{\eta\eta}}\boldsymbol{u} + \boldsymbol{T}_2(\mathrm{D}\boldsymbol{u})\partial_{\boldsymbol{\xi\xi}}\boldsymbol{u} + \boldsymbol{T}_3(\mathrm{D}\boldsymbol{u})\partial_{\boldsymbol{\eta\xi}}\boldsymbol{u} \tag{3}$$

where $\boldsymbol{\eta}$ *is the major, and* $\boldsymbol{\xi}$ *the minor eigenvector of the structure tensor* $\boldsymbol{J} := \boldsymbol{J}(\mathrm{D}\boldsymbol{u}) := \boldsymbol{\nabla} u \boldsymbol{\nabla} u^{\mathrm{T}} + \boldsymbol{\nabla} v \boldsymbol{\nabla} v^{\mathrm{T}} = \mathrm{D}\boldsymbol{u}^{\mathrm{T}}\mathrm{D}\boldsymbol{u}$. *The coefficient matrices* $\boldsymbol{T}_i(\mathrm{D}\boldsymbol{u})$, *are given by*

$$\boldsymbol{T}_i(\mathrm{D}\boldsymbol{u}) := \boldsymbol{R}\,\boldsymbol{\Theta}_i\big(|\partial_{\boldsymbol{\eta}}\boldsymbol{u}|, |\partial_{\boldsymbol{\xi}}\boldsymbol{u}|\big)\,\boldsymbol{R}^{\mathrm{T}}\,, \qquad i = 1, 2, 3\,, \tag{4}$$

$$\boldsymbol{\Theta}_1(r, s) := \mathrm{diag}\big(\vartheta_1(r), \vartheta_2(r, s)\big)\,, \qquad \vartheta_1(z) := \frac{1 - 8\beta^2 z^2}{(1 + \beta^2 z^2)^2}\,, \tag{5}$$

$$\boldsymbol{\Theta}_2(r, s) := \mathrm{diag}\big(\vartheta_2(r, s), \vartheta_1(s)\big)\,, \qquad \vartheta_2(w, z) := \frac{3}{(1 + \beta^2 w^2)(1 + \beta^2 z^2)}\,, \tag{6}$$

$$\boldsymbol{\Theta}_3(r, s) := -2\begin{pmatrix} 0 & \vartheta_3(r, s) \\ \vartheta_3(s, r) & 0 \end{pmatrix}, \quad \vartheta_3(w, z) := \frac{w}{z}\,\frac{1 + 4\beta^2 z^2}{(1{+}\beta^2 w^2)(1{+}\beta^2 z^2)}\,, \tag{7}$$

where $\boldsymbol{R} = (\mathrm{D}\boldsymbol{u}^{-1})^{\mathrm{T}}\boldsymbol{P}\mathrm{diag}\big(|\partial_{\boldsymbol{\eta}}\boldsymbol{u}|, |\partial_{\boldsymbol{\xi}}\boldsymbol{u}|\big)$ *is a rotation matrix that aligns the principal components of the variation of* $\boldsymbol{u}$ *with the coordinate axes in both the* (x, y) *and* (u, v) *spaces;* $\boldsymbol{P} = (\boldsymbol{\eta} \mid \boldsymbol{\xi})$ *is the eigenvector matrix of* $\boldsymbol{J}$.

The proof of this theorem relies on two principles that allow to reduce the general situation of the theorem to a sequence of more specialised cases, which are successively treated in three lemmas. The first of these principles has been established in [25] where univariate amoeba median filtering was considered in a setting where a function u was filtered using amoebas computed from a different function f. Then u influences the filter result by the curvature of its level lines, whereas f exerts its influence via the asymmetric shape of the amoeba. The decomposition of the effect of the amoeba median filter into a curvature-related and an asymmetry-related part remains valid even when both functions are the same, and we will use it in the proof of Lemma 3. The second principle is to use invariances of the amoeba construction and the median filter to normalise the local geometry of a configuration. This was done in a non-adaptive setting for the L^1 median in [28] using Euclidean transformations, whereas [27] used affine transformations to normalise the geometry for affine equivariant medians. In our context, the amoeba construction as well as the median filters are Euclidean equivariant which allows to align the function by rotations to one with diagonal Jacobian, see Lemma 3. Deriving in this setting the amoeba shape from the bivariate function and treating further the function being filtered and the amoeba separately, according to the first principle above, allows to exploit the affine equivariance of the median filter for a further normalisation, making the Jacobian of the function a unit matrix, see Lemmas 1 and 2. On this level, the result for non-adaptive bivariate median filtering from [27] can be invoked for the curvature contribution, Lemma 1, whereas the asymmetry contribution is accessible to direct analysis via the definition of the Oja median, Lemma 2.

Lemma 1. *(from [27], Lemma 2 there) Let* $\boldsymbol{u}$ *be given as in Theorem 1 with the origin* $\mathbf{0} = (0, 0)$ *in the interior of* Ω*. Assume that the Jacobian* $\mathrm{D}\boldsymbol{u}(\mathbf{0})$ *is the* 2×2 *unit matrix* $\boldsymbol{I}$*, i.e.* $u_x = v_y = 1$*,* $u_y = v_x = 0$*. One step of Oja median filtering of* $\boldsymbol{u}$ *at* $\mathbf{0}$ *with the disc* D_ϱ *of radius* ϱ *as (non-adaptive) structuring element approximates an explicit time step of size* $\tau = \varrho^2/24$ *of the PDE system*

$$u_t = u_{xx} + 3u_{yy} - 2v_{xy} \; , \qquad v_t = 3v_{xx} + v_{yy} - 2u_{xy} \; . \tag{8}$$

The second lemma refers to the filtering of a linear bivariate function with amoeba structuring elements computed from another function.

Lemma 2. *Let* $\boldsymbol{u}$ *be given as in Theorem 1 with* $\mathbf{0}$ *in the interior of* Ω*, and* $\mathrm{D}\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{I}$ *for all* $\boldsymbol{x} \in \Omega$*. At* $\mathbf{0}$*, let an amoeba structuring element* $\mathcal{A}(\mathbf{0})$ *be given in polar coordinates* (r, φ) *with* $x = r\cos\varphi$*,* $y = r\sin\varphi$ *by its contour* $r(\varphi) = \varrho - a(\varphi)$*,* $a(\varphi) := \frac{1}{2}\varrho^2\beta^2(\alpha_1\cos^3\varphi + \alpha_2\cos^2\varphi\sin\varphi + \alpha_3\cos\varphi\sin^2\varphi + \alpha_4\sin^3\varphi)$*. Then one step of Oja median filtering of* $\boldsymbol{u}$ *at* $\mathbf{0}$ *approximates an explicit time step of size* $\tau = \varrho^2/24$ *of the PDE system*

$$u_t = -9\beta^2\alpha_1 - 3\beta^2\alpha_3 \; , \qquad v_t = -3\beta^2\alpha_2 - 9\beta^2\alpha_4 \; . \tag{9}$$

Proof (of Lemma 2). Due to the linearity of $\boldsymbol{u}$, the Oja median of $\boldsymbol{u}$ within the disc $D_\varrho(\mathbf{0})$ with contour $r(\varphi) = \varrho$ equals $\boldsymbol{u}(\mathbf{0})$. We study how the perturbation of the amoeba contour $r(\varphi)$ by $-a(\varphi)$ changes the gradient $\boldsymbol{g}$ of the objective function of (2) at $\mathbf{0}$. Remembering that the Oja median is the minimiser for a sum of triangle areas in the u-v plane, this means that we are interested in the net contribution of those triangles which are added or removed when switching from the disc $D_\varrho(\mathbf{0})$ to the actual amoeba. Neglecting a higher order error, the added or removed areas can be projected to the circumference of the disc. We are therefore led to consider the gradients $\boldsymbol{g}_{B,A;M}$ of the areas of triangles MAB w.r.t. M, where M is the candidate median point, $A \equiv A(\varphi) = (\varrho\cos\varphi, \varrho\sin\varphi)$ and B in the disc D_ϱ. Up to higher order terms, $\boldsymbol{g}$ will be the resultant of $\boldsymbol{g}_{B,A;M}$ for all such triangles, each weighted with $-a(\varphi)$ and evaluated at $M = \mathbf{0}$. Note that the weights $-a$ can take either sign, depending on whether the amoeba boundary is inside or outside the disc near A.

For $M = \mathbf{0}$ the straight line MA bisects D_ϱ; thus, for each point $B \in D_\varrho$ also the point B' obtained by reflecting B on MA is in D_ϱ. Thus, the gradients of the triangle areas of MAB and MAB' (w.r.t. M) can be combined into the gradient $\boldsymbol{g}_{B,A,B';M}$ of the area of the quadrangle $MB'AB$ (a kite, a.k.a. deltoid). The vector $\boldsymbol{g}_{B,A,B';M}$ is perpendicular to BB' and proportional to $\frac{1}{2}|BB'|$, the height of B over the line MA. Thus, $\boldsymbol{g}_{B,A,B';M} = -\frac{1}{2}|BB'|(cos\varphi, \sin\varphi)$. Integration over B in a half-disc yields the aggregated gradient for all triangles MAB with the point A as $\boldsymbol{g}_{A;M} = -\frac{2}{3}\varrho^3(\cos\varphi, \sin\varphi)$. Integrating over directions φ with weights $-a(\varphi)$ yields

$$\boldsymbol{g} = \int_0^{2\pi} -a_\varphi \boldsymbol{g}_{A(\varphi);M}\,\mathrm{d}\varphi = -\frac{\varrho^6\beta^2\pi}{12}\begin{pmatrix}3\alpha_1+\alpha_3\\ \alpha_2+3\alpha_4\end{pmatrix}\,. \tag{10}$$

From the proof of Lemma 2 in [27] (i.e. Lemma 1 above) it can be read off that a gradient $\boldsymbol{g}$ for (2) is compensated by a shift of M by $\boldsymbol{g}/(-\frac{2}{3}\pi\varrho^4)$ which means that the median of $\boldsymbol{u}$ within the amoeba $\mathcal{A}(\mathbf{0})$ amounts to $\boldsymbol{u}(\mathbf{0}) - \beta^2\begin{pmatrix}3\alpha_1+\alpha_3\\ \alpha_2+3\alpha_4\end{pmatrix}$, which is exactly the time step for the PDE system (9) as claimed in the lemma.

Lemma 3. *Let $\boldsymbol{u}$ be given as in Theorem 1 with the origin $\mathbf{0} = (0,0)$ in the interior of Ω. Assume that the Jacobian $\mathrm{D}\boldsymbol{u}(\mathbf{0})$ is diagonal, with $u_x \ge v_y > 0$, $u_y = v_x = 0$. Then one step of amoeba median filtering with the Oja median at $\mathbf{0}$ approximates for $\varrho \to 0$ an explicit time step of size $\tau = \varrho^2/24$ of the PDE system*

$$u_t = \vartheta_1(u_x)\,u_{xx} + \vartheta_2(u_x,v_y)\,u_{yy} - 2\vartheta_3(u_x,v_y)\,v_{xy}\ , \tag{11}$$

$$v_t = \vartheta_2(u_x,v_y)\,v_{xx} + \vartheta_1(v_y)\,v_{yy} - 2\vartheta_3(v_y,u_x)\,u_{xy}\ , \tag{12}$$

with $\vartheta_{1,2,3}$ as in Theorem 1.

Proof (of Lemma 3). In [28, Sec. 4.1.2] the effect of an univariate amoeba median filter step was analysed into two parts, a *curvature contribution* coming from the curvature of level lines of the function being filtered, and an *asymmetry contribution* reflecting the asymmetry of the amoeba. Interactions of the two manifest only in higher order terms w.r.t. ϱ that can safely be neglected in the PDE analysis. The same decomposition can also be applied here. We denote by $\boldsymbol{f}$ the function obtained from $\boldsymbol{u}$ by linearisation at $\mathbf{0}$. Then the curvature contribution at $\mathbf{0}$ is equivalent to a step of amoeba median filtering of the function $\boldsymbol{u}$ with the amoeba $\mathcal{A}_{\boldsymbol{f}}(\mathbf{0})$ computed from $\boldsymbol{f}$, whereas for the asymmetry contribution $\boldsymbol{f}$ is filtered using the amoeba $\mathcal{A}_{\boldsymbol{u}}(\mathbf{0})$ computed from $\boldsymbol{u}$.

To start with the curvature contribution, $\mathcal{A}_f(\mathbf{0})$ is an ellipse with half-axes ϱ/U in x direction, and ϱ/V in y direction, with $U := \sqrt{1+\beta^2u_x^2}$, $V := \sqrt{1+\beta^2v_y^2}$. By the affine equivariance of the Oja median, we can apply first to the x-y plane an affine transform T_{xy} with the transformation matrix $\boldsymbol{M} := \mathrm{diag}(U,V)$, so the amoeba is turned into the disc D_ϱ, and the image into $\tilde{\boldsymbol{u}}$ with Jacobian $\mathrm{D}\tilde{\boldsymbol{u}} = \mathrm{D}\boldsymbol{u}\,\boldsymbol{M}^{-1}$. Second, the u-v plane is transformed by an affine transform T_{uv} with the transformation matrix $\mathrm{D}\tilde{\boldsymbol{u}}^{-1}$ which yields a bivariate image $\hat{\boldsymbol{u}}$ with unit Jacobian. Now the hypotheses of Lemma 1 are satisfied for $\hat{\boldsymbol{u}}$. Using (8) for $\hat{\boldsymbol{u}}$ and reverting T_{uv} and T_{xy} yields the curvature contribution

$$u_t^{\mathrm{curv}} = \frac{u_{xx}}{U^2} + \frac{3u_{yy}}{V^2} - \frac{2u_xv_{xy}}{v_yU^2}\ , \qquad v_t^{\mathrm{curv}} = \frac{3v_{xx}}{U^2} + \frac{v_{yy}}{V^2} - \frac{2v_yu_{xy}}{u_xV^2}\ . \tag{13}$$

Turning to the asymmetry contribution, we compute the contour of $\mathcal{A}_{\boldsymbol{u}}(\mathbf{0})$ in a way that after applying the same two affine transformations as before suits the hypothesis of Lemma 2. To this end, consider the vector $\boldsymbol{w}^* := (U\cos\varphi, V\sin\varphi)$, with U and V as above, and normalise it to $\boldsymbol{w} := \boldsymbol{w}^*/\|\boldsymbol{w}^*\|$. Then the directional derivatives of $\boldsymbol{u}$ become $\partial_{\boldsymbol{w}}\boldsymbol{u} = (u_x V\cos\varphi/G, v_y U\sin\varphi/G)$, where $G := \sqrt{V^2\cos^2\varphi + U^2\sin^2\varphi}$, and $\partial_{\boldsymbol{ww}}\boldsymbol{u} = \frac{V^2}{G^2}\cos^2\varphi\partial_{xx}\boldsymbol{u} + \frac{2UV}{G^2}\cos\varphi\sin\varphi\partial_{xy}\boldsymbol{u} + \frac{U^2}{G^2}\sin^2\varphi\partial_{yy}\boldsymbol{u}$. The point of the amoeba contour in direction $\boldsymbol{w}$ is given up to higher order terms by $r\boldsymbol{w}$ where $r \equiv r(\boldsymbol{w})$ satisfies the condition

$$\varrho = \int_0^r \sqrt{1 + \beta^2|\partial_{\boldsymbol{w}}\boldsymbol{u} + s\partial_{\boldsymbol{ww}}\boldsymbol{u}|^2}\,\mathrm{d}s\ . \tag{14}$$

The r.h.s. of this equation is the length of a straight line in direction $\boldsymbol{w}$ under the Euclidean amoeba metric. Note that the actual shortest path from $\mathbf{0}$ to the amoeba contour under the amoeba metric can deviate from this line but as pointed out in [28, 4.1.2] this deviation only influences higher order terms. Again up to higher order terms (14) can be evaluated to

$$\varrho = r\sqrt{1+\beta^2|\partial_{\boldsymbol{w}}\boldsymbol{u}|^2}\left(1 + r\frac{\beta^2\langle\partial_{\boldsymbol{w}}\boldsymbol{u}, \partial_{\boldsymbol{ww}}\boldsymbol{u}\rangle}{2(1+\beta^2|\partial_{\boldsymbol{w}}\boldsymbol{u}|^2)}\right)\ , \tag{15}$$

$$r = \frac{\varrho}{\sqrt{1+\beta^2|\partial_{\boldsymbol{w}}\boldsymbol{u}|^2}}\left(1 - \frac{\varrho\beta^2\langle\partial_{\boldsymbol{w}}\boldsymbol{u}, \partial_{\boldsymbol{ww}}\boldsymbol{u}\rangle}{2(1+\beta^2|\partial_{\boldsymbol{w}}\boldsymbol{u}|^2)^{3/2}}\right)\ . \tag{16}$$

Applying the affine transform T_{xy} as above, $\boldsymbol{w}^*$ becomes the vector $(\cos\varphi, \sin\varphi)$, and $r(\boldsymbol{w})$ is transformed into $r(\varphi)$ as in the hypothesis of Lemma 2 with

$$\alpha_1 = \frac{u_x u_{xx}}{U^3}\ , \qquad \alpha_2 = \frac{2u_x u_{xy} + v_y v_{xx}}{U^2 V}\ , \tag{17}$$

$$\alpha_3 = \frac{u_x u_{yy} + 2v_y v_{xy}}{UV^2}\ , \qquad \alpha_4 = \frac{v_y v_{yy}}{V^3}\ . \tag{18}$$

The derivatives u_x etc. herein refer to the untransformed function $\boldsymbol{u}$. Applying further T_{uv} as above, the setting is fully adapted to the hypothesis of Lemma 2. Reverting T_{uv} and T_{xy} on (9), then inserting (17), (18) yields the desired asymmetry contribution as

$$u_t^{\mathrm{asym}} = -9\beta^2\frac{u_x^2 u_{xx}}{U^4} - 3\beta^2\frac{u_x^2 u_{yy}}{U^2V^2} - 6\beta^2\frac{u_x v_y v_{xy}}{U^2V^2}\ , \tag{19}$$

$$v_t^{\mathrm{asym}} = -3\beta^2\frac{v_y^2 v_{xx}}{U^2V^2} - 9\beta^2\frac{v_y^2 v_{yy}}{V^4} - 6\beta^2\frac{u_x v_y u_{xy}}{U^2V^2}\ . \tag{20}$$

Combining (13) and (19), (20) as $\boldsymbol{u}_t := \boldsymbol{u}_t^{\mathrm{curv}} + \boldsymbol{u}_t^{\mathrm{asym}}$ yields (11), (12).

Proof (of Theorem 1). Consider a bivariate image $\boldsymbol{u}$ with regular Jacobian. By translation invariance, it can be assumed that the filter is considered at location $\mathbf{0}$. As eigenvectors of the structure tensor, $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ are orthonormal. Moreover, the directional derivatives of $\boldsymbol{u}$ in these directions, i.e. $\mathrm{D}\boldsymbol{u}\cdot\boldsymbol{\eta}$ and $\mathrm{D}\boldsymbol{u}\cdot\boldsymbol{\xi}$, are orthogonal. Thus both the x-y plane and the u-v plane can be rotated in order to align these orthogonal pairs with the respective coordinate axes. For the x-y plane, this is achieved using the rotation matrix $\boldsymbol{P}$; for the u-v plane, taking into account the non-unit lengths of the two orthogonal vectors, one obtains $\boldsymbol{R}^{-1}$ as the appropriate rotation matrix. Applying both rotations yields a bivariate image that satisfies the conditions of Lemma 3. Applying the inverse rotations to (11), (12) yields the PDE system stated in (3)–(7).

We turn now to the second affine equivariant median filter under consideration, the transformation–retransformation L^1 median. As in the case of non-adaptive filtering [27], we find that it approximates the same PDE as the Oja median filter.

Theorem 2. *Let a bivariate function $\boldsymbol{u} : \Omega \to \mathbb{R}^2$ with $\mathrm{D}\boldsymbol{u}$ of rank 2 be given as in Theorem 1. For $\varrho \to 0$, one step of amoeba median filtering with the transformation–retransformation L^1 median approximates a time step of size $\tau = \varrho^2/24$ of the same PDE* (3)–(7) *as for the Oja median.*

Proof (of Theorem 2). We follow the same strategy as for Theorem 1. Euclidean and affine transformations can again be used to reduce the generic geometric setting to the hypotheses of Lemma 3 and further to the hypotheses of Lemmas 1 and 2. As pointed out in [27, Sec. 3.1.5], this reduction procedure reproduces the transformations of the transformation–retransformation L^1 median. Thus, it remains to secure statements analogous to Lemmas 1 and 2 for the standard L^1 median. As for Lemma 1, this is a special case of Lemma 1 from [27]. To prove the L^1 analogue of Lemma 2 one calculates the gradient of the objective function of (1), which is even easier than for the Oja median because, adopting notations from the proof of Lemma 2, said gradient is composed just of the vectors $MA = (\cos\varphi, \sin\varphi)$ with appropriate weighting.

Remark 1. In the univariate case an interesting relation holds between the PDEs associated with non-adaptive and adaptive median filters: The self-snakes PDE [18] $u_t = |\boldsymbol{\nabla} u| \operatorname{div}\bigl(g(|\boldsymbol{\nabla} u|)\boldsymbol{\nabla} u/|\boldsymbol{\nabla} u|\bigr)$ – the limit case of the amoeba filter – is obtained from the curvature motion PDE $u_t = |\boldsymbol{\nabla} u| \operatorname{div}\bigl(\boldsymbol{\nabla} u/|\boldsymbol{\nabla} u|\bigr)$ – the limit case of the nonadaptive filter – by inserting the edge-stopping function g within the divergence expression.

We have to defer a more detailed discussion of the PDE system (3)–(7) to future work. In this context, an interesting question to be answered will be whether a relation similar to that between univariate self-snakes and curvature motion can be established between (3)–(7) and the PDE system derived in [27] for the non-adaptive case.

5 Experiments

We validate our PDE approximation results in the axis-aligned setting of Lemma 3 for eight simple bivariate example functions $\boldsymbol{u} = (u, v)$ given by $u(x, y) = u_x x + \frac{1}{2}u_{xx}x^2 + u_{xy}xy + \frac{1}{2}u_{yy}y^2$, $v(x, y) = v_y y + \frac{1}{2}v_{xx}x^2 + v_{xy}xy + \frac{1}{2}v_{yy}y^2$, where all derivatives refer to the location $(0, 0)$. The example functions are collected in Table 1.

For the functions $\boldsymbol{u}$ and their linearisations $\boldsymbol{f}$, we compute approximations of amoeba structuring elements $\mathcal{A}^{\boldsymbol{u}}_\varrho(\mathbf{0})$ and $\mathcal{A}^{\boldsymbol{f}}_\varrho(\mathbf{0})$ with contrast scale $\beta = 1$ and amoeba radius $\varrho = 1$ on a discrete Cartesian grid with sample distance $h = 0.015$. For good approximation of rotational invariance, we base our amoeba computation on neighbourhoods of radius $h\sqrt{10}$, which include 36 points instead of 4- or 8-neighbourhoods. (This makes sense for our fairly smooth example functions; in an image filtering application, however, the use of such large neighbourhoods would spoil the adaptation of amoebas to fine image details.) The sizes of these amoebas are also given in Table 1.

The Oja and transformation–retransformation L^1 medians of $\boldsymbol{u}$ within $\mathcal{A}_\varrho^{\boldsymbol{f}}(\mathbf{0})$ are then computed by gradient descent methods, see [27], and compared with the isolated curvature contribution (13). In the same way, the medians of $\boldsymbol{f}$ within $\mathcal{A}_\varrho^{\boldsymbol{u}}(\mathbf{0})$ are computed and compared with the isolated asymmetry contribution (19), (20), and the medians of $\boldsymbol{u}$ within $\mathcal{A}_\varrho^{\boldsymbol{u}}(\mathbf{0})$ with the full PDE system (11), (12). In Table 2 the results of median computations and the corresponding time steps with size $\tau = \varrho^2/24$ are given.

As can be seen, the results of both median filters agree well with the PDE time steps for the first six example functions a–f in which only one second-order coefficient is nonzero, with deviations in the order of 10^{-4}. In the last two examples g, h where all six second-order coefficients are nonzero, there are deviations in the order of 10^{-3} for individual coordinates. However, these deviations are well within the range to be expected for the given grid discretisation.

6 Summary and Outlook

We have derived a PDE approximated by amoeba median filtering of bivariate images with the affine equivariant Oja median or transformation–retransformation L^1 median, for Euclidean amoeba metrics. Numerical computations on sampled test functions confirmed the approximation. A more detailed discussion of the PDE will be a subject of forthcoming work where also the degenerate case $\det \mathrm{D}\boldsymbol{u} = 0$ that we excluded here (as before in [27]) should be investigated.

Our hope is that the result presented will contribute to a deeper understanding of adaptive filtering and the relations between morphological and PDE-based image filters. We are optimistic that we will be able in our ongoing work to extend the result to practically more meaningful cases such as three-channel planar or volume images, such as done in [27] for non-adaptive median filters.

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Table 1. Bivariate example functions $\boldsymbol{u}$ for the validation of PDE approximations, and sizes (in pixels) of the amoeba approximations.

Test case	$u(x,y)$	$v(x,y)$	$\lvert\mathcal{A}^{\boldsymbol{u}}_{\varrho}(\mathbf{0})\rvert$	$\lvert\mathcal{A}^{\boldsymbol{f}}_{\varrho}(\mathbf{0})\rvert$
a	$x+0.15\,x^2$	y	6992	6977
b	$x+0.3\,xy$	y	6995	6977
c	$x+0.15\,y^2$	y	6973	6977
d	$2x+0.15\,x^2$	$0.5\,y$	5584	5573
e	$2x+0.3\,xy$	$0.5\,y$	5583	5573
f	$2x+0.15\,y^2$	$0.5\,y$	5571	5573
g	$2x-0.15\,x^2+0.1\,xy+0.1\,y^2$	$0.5\,y+0.05\,x^2+0.3\,xy-0.1\,y^2$	5574	5573
h	$2x-0.25\,x^2+0.2\,xy+0.15\,y^2$	$0.5\,y+0.05\,x^2+0.6\,xy-0.15\,y^2$	5556	5573

Table 2. Amoeba median values and time steps of approximated PDEs for the example functions $\boldsymbol{u}$ from Table 1. All values are multiplied by 10^4.

Test case	Coordinate	Curvature contribution			Asymmetry contribution			Complete filter		
		TRL^1	Oja	PDE	TRL^1	Oja	PDE	TRL^1	Oja	PDE
a	u	62	63	63	−281	−282	−281	−216	−216	−219
	v	0	0	0	0	0	0	0	0	0
b	u	0	0	0	0	0	0	0	0	0
	v	−125	−125	−125	−187	−187	−188	−312	−311	−313
c	u	187	187	188	−90	−90	−94	94	97	94
	v	0	0	0	0	0	0	0	0	0
d	u	24	24	25	−183	−182	−180	−158	−155	−155
	v	0	0	0	0	0	0	0	0	0
e	u	0	0	0	0	0	0	0	0	0
	v	−50	−50	−50	−116	−116	−120	−166	−167	−170
f	u	300	299	300	−237	−237	−240	62	62	60
	v	0	0	0	0	0	0	0	0	0
g	u	−23	−24	−25	−95	−96	−100	−122	−121	−125
	v	−61	−59	−59	74	73	75	14	14	17
h	u	−128	−133	−142	−159	−159	−180	−304	−301	−322
	v	−118	−108	−108	75	75	95	−37	−35	−13

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